The Eigenvalue Problem for Solitary Waves of the Green-Naghdi Equations

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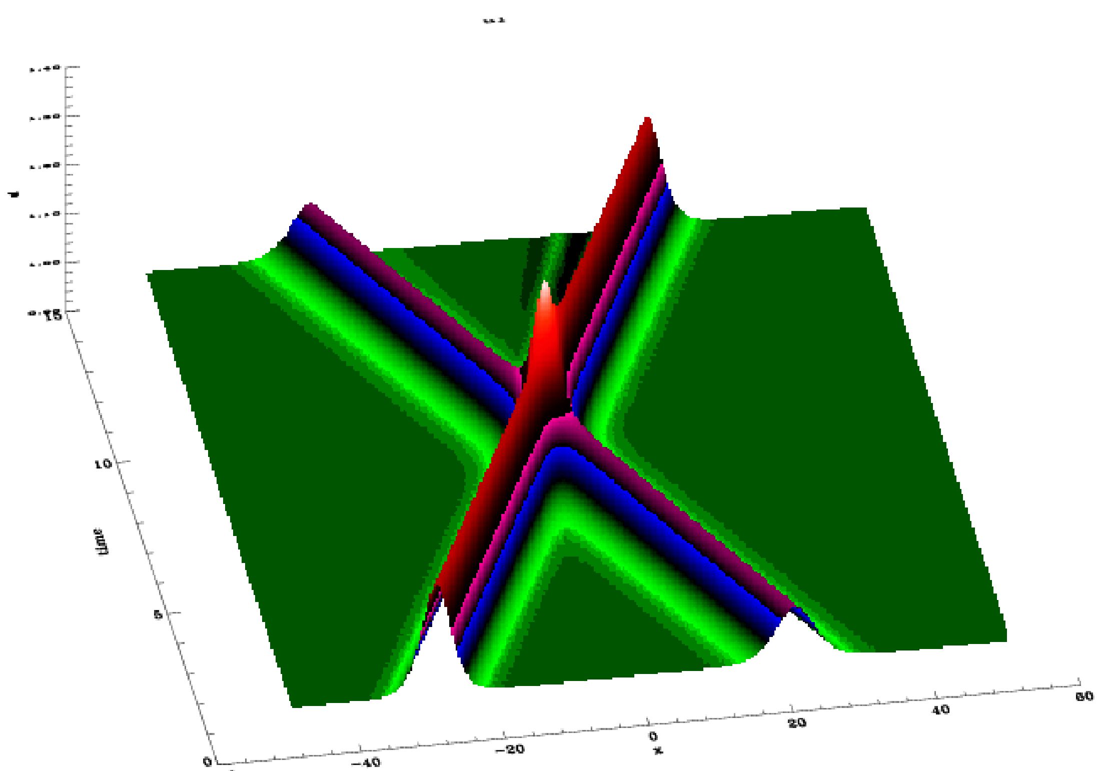
Abstract

We investigate the eigenvalue problem obtained from linearization of nonlinearly dispersive evolution equations about solitary wave solutions using the technique of the Evans function. Different from weakly nonlinear water wave models, the physical system considered here has nonlinearity in its highest derivative term. This results in a more detailed asymptotic analysis of the eigenvalue problem in the presence of a large parameter. Combining the technique of singular perturbation with the Evans function, we show that the problem has no eigenvalues of positive real part and the Evans function is non-vanishing everywhere except the origin.



Solitary wave interaction of GN equations.

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1 Introduction

It has long been an issue to model and understand the full water wave problem due to its broad applications to coastal engineering, prediction of ocean climate, and fluid mechanics. The full water wave problem is imposed as a fully nonlinear system. Although a great deal of effort has been made to directly tackle it both numerically and analytically, the problem is still not completely well understood due to the complexity of its nonlinearity. Another way to deal with this problem is to develop approximate models under a variety of physical conditions. One primary approach was linear approximation under the assumption of a small perturbation from a quiescent state. While using a higher order approximation, weakly nonlinear models have been developed in the parameter regime of small amplitude and long wave length. Among them are the well-known Korteweg-de Vries (KdV) and Boussinesq equations [15]. The acquisition of these equations confirmed the existence of solitary waves for the the full water wave problem, as a consequence, leading to the development of theories on solitons, integrability and inverse scattering transform [1]. Despite both the physical and mathematical importance of the weakly nonlinear approximation for the full water wave problem, it has limitations to model higher nonlinear phenomena, including high-amplitude waves and wave breaking. Efforts have been made to obtain higher nonlinear model equations. Among them, the Green-Naghdi equations (rf. [9], [10], [6], [5])

$$
\begin{aligned}
&\eta_t + (u\eta)_x = 0,\\
&u_t + uu_x + \eta_x = \frac{1}{3\eta}\big(\eta^2 \frac{d}{dt}(\eta u_x)\big)_x,
\end{aligned}
$$

were derived for both free surface and inter-facial surface waves under the assumption of long wave length but relatively large amplitude compared with the depth of the fluid. Here, $\frac{d}{dt}(\eta u_x) = (\eta u_x)_t + u(\eta u_x)_x$, η and u represent the surface disturbance and the horizontal velocity, respectively. Numerical comparisons [6] have shown that the Green-Naghdi equations have a broader parameter regime to approximate the full water wave problem, especially, in the regime of relatively large amplitude. In this paper, we shall analytically investigate stability of solitary waves of the Green-Naghdi equations. It is well-known that solitary waves of the KdV equation are orbitally stable and this has been proved by using a variational method due to the fact that its solitary waves are minimizers of its Hamiltonian functional [4]. However, this is not a common property for the full water wave problem, a class of Boussinesq equations [14], and the Green Naghdi equations. As a matter of fact, the second variation of their Hamiltonian functional subject to certain constraints are indefinite. Therefore, variational approach for the stability analysis of these systems may not be applicable. In this paper, we adapt the other technique, the Evans function, to investigate eigenvalue problems for solitary waves of the Green-Naghdi equations. Evans [8] has used this method for the stability issue of the impulses in nerve axon equation. Later, this method was further developed by Jones *et al,* [2], [3] [12], and Pego and Weinstein [13], [14] to apply it to a wide range of nonlinear evolution equations, including the weakly nonlinear KdV equation and Boussinesq equations. However, compared with weakly nonlinear models, the higher nonlinearity possessed by the Green-Naghdi equations is more challenging to consider stability of their solitary waves and demands more detailed analysis on this system. In this paper, we shall use singular perturbation theory to deal with the case when the eigenvalue problem for linear stability analysis has a large parameter. In Section 2, we study the Hamiltonian structure of the Green-Naghdi Equations. In Section 3, we shall derive the eigenvalue problem for solitary waves and discuss its properties. The Section 4, we use the singular perturbation method to show that there are no eigenvalues of large magnitude at least when the speed of propagation of solitary waves are close to that of linear waves, and thus the Evans function does not have zeros in a neighbourhood of infinity. In the last section, we shall use the KdV approximation to show that the Evans function has only one zero at the origin and thus the corresponding eigenvalue problems has no eigenvalue problem of positive real part.

2 The Hamiltonian Structure of the Green-Naghdi Equations

Assume that the depth of the fluid flow is h so that the surface disturbance η and the horizontal velocity u satisfy the condition $\eta \to h$ and $u \to 0$ as $|x| \to \infty$.

To transform the Green Naghdi equations to a non-dimensional system, we let $x = h\tilde{x}$, $t = \sqrt{h/g}\,\tilde{t}$, $\eta = h\tilde{\eta}$ and $u = \sqrt{hg}\,\tilde{u}$. For the sake of simplicity, we also omit using the accent ˜ in the rest of the paper. Then we obtain the following dimensionless system,

$$\begin{aligned} &\eta_t + (u\eta)_x = 0, \\ &u_t + uu_x + \eta_x = \frac{1}{3\eta}\big(\eta^2 \frac{d}{dt}(\eta u_x)\big)_x, \end{aligned}$$

such that $\eta \to 1$ as $|x| \to \infty$. One may also multiply the first equation by u and the second equation by η, and then adding the resulting equations together. By letting $w = u\eta$, we obtain the equivalent system

$$\begin{aligned} &\eta_t + w_x = 0, \\ &w_t = -(\frac{w^2}{\eta})_x - \eta\eta_x + \frac{1}{3}\left(\eta^2 \frac{d}{dt}(\eta(\frac{w}{\eta})_x)\right)_x . \end{aligned} \tag{2.1}$$

The system (2.1) will be used to conduct linear stability analysis of solitary waves in this paper.

The Green Naghdi equations have a Hamiltonian structure of the form

$$\begin{pmatrix} w_t \\ \eta_t \end{pmatrix} = J \begin{pmatrix} \frac{\delta H}{\delta w} \\ \frac{\delta H}{\delta \eta} \end{pmatrix},$$

where the Hamiltonian functional H takes the form

$$H = \frac{1}{2}\int \left(\frac{w^2}{\eta} + \frac{1}{3}\eta^3 (\frac{w}{\eta})_x^2 + (\eta - h)^2 \right) dx,$$

and the Hamiltonian operator J may be expressed as the product of three 2x2 matrix operators $J = B^{-1}\tilde{J}(B^*)^{-1}$. Here

$$\tilde{J} = -\begin{pmatrix} \partial m + m\partial & \eta\partial \\ \partial\eta & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} I - \frac{1}{3}\partial\eta^3\partial\frac{1}{\eta} & -\partial(\frac{w}{\eta})_x\eta^2 + \frac{1}{3}\partial\eta^3\partial\frac{w}{\eta^2} \\ 0 & I \end{pmatrix},$$

$m = w - \frac{1}{3}\big(\eta^3(\frac{w}{\eta})_x\big)_x$ and ∂ is the derivative with respect to the spatial variable x. Under the assumption that η is positive and $\eta \to 1$ as $|x| \to \infty$, the operator $\mathcal{L} = I - \frac{1}{3}\partial\eta^3\partial\frac{1}{\eta}$ is a positive, sturm-liouville operator and thus has a bounded inverse $\mathcal{L}^{-1}$. Therefore, B and its adjoint operator B^* are also invertible, denoted by B^{-1} and $(B^*)^{-1}$, respectively. Another conserved quantity of this system Q, so called the conservation of momentum, is

$$Q = \int m(1 - \frac{1}{\eta})dx.$$

Hamiltonian structures and conserved quantities of evolution equations have been used to study stability of their travelling wave solutions. However, this method may not apply to the stability analysis of solitary waves of the Green Naghdi equations directly. A solitary wave solution $(w(x-ct), \eta(x-ct))$ of the system (2.1) takes the form of sech-functions such that

$$\begin{aligned} w &= c(c^2-1)\,\mathrm{sech}\big(\frac{\sqrt{3(c^2-1)}\,(x-ct)}{2c}\big)^2, \\ \eta &= 1 + \frac{w}{c} \end{aligned}$$

for any fixed constant c with $|c| > 1$. A direct calculation shows that each solitary wave solution $(w(x-ct), \eta(x-ct))$ is a critical point of the functional $H - c\,Q$, *i.e.* the identities

$$\frac{\delta H}{\delta w} - c\frac{\delta Q}{\delta w} = 0, \qquad \frac{\delta H}{\delta \eta} - c\frac{\delta Q}{\delta \eta} = 0,$$

hold at the solitary wave. Taking the second variations of the functionals H and Q, we obtain

$$H'' = \begin{pmatrix} (1/\eta)\mathcal{L} & C \\ C^* & D \end{pmatrix}, \qquad Q'' = \begin{pmatrix} 0 & E \\ E^* & F \end{pmatrix},$$

with operators

$$\begin{aligned} C &= -\frac{m}{\eta^2} - \frac{1}{\eta}\partial\left((\frac{w}{\eta})_x \eta^2 - \frac{1}{3}\eta^3 \partial \frac{w}{\eta^2}\right), \\ D &= I + \frac{2wm}{\eta^3} - \frac{w^2}{\eta^3} - \eta^2(\frac{w}{\eta})_x \partial \frac{w}{\eta^2} + \frac{w}{\eta^2}\partial(\frac{w}{\eta})_x\eta^2 - \frac{w}{3\,\eta^2}\partial\eta^3\partial\frac{w}{\eta^2}, \\ E &= \mathcal{L}^*\frac{1}{\eta^2} + \frac{1}{3\,\eta^2}(\eta\eta_x)_x - \frac{1}{\eta}\partial\eta_x, \\ F &= -\frac{2m}{\eta^3} + \frac{2\eta_x}{\eta}(\frac{w}{\eta})_x - \frac{2w}{3\eta^3}(\eta\eta_x)_x - \eta_x\partial\frac{w}{\eta^2} + \frac{w}{\eta^2}\partial\eta_x \\ &\quad - \frac{1}{\eta^2}\partial(\frac{w}{\eta})_x\eta^2 + \eta^2(\frac{w}{\eta})_x\partial\frac{1}{\eta^2} + \frac{1}{3\eta^2}\partial\eta^3\partial\frac{w}{\eta^2} + \frac{w}{3\eta^2}\partial\eta^3\partial\frac{1}{\eta^2}. \end{aligned}$$

It follows from Weyl's essential spectrum theorem that the essential spectrum of the operator $H'' - c\,Q''$ evaluated at a solitary wave coincides with that of its asymptotic operator $H''_\infty - c\,Q''_\infty$ as $|x| \to \infty$. Since

$$H''_\infty - c\,Q''_\infty = \begin{pmatrix} I - \frac{1}{3}\partial^2 & -c(I - \frac{1}{3}\partial^2) \\ -c(I - \frac{1}{3}\partial^2) & I \end{pmatrix},$$

it follows that the essential spectrum consists of the intervals $(-\infty, 1-|c|\,]$ and $[\,1+|c|, \infty)$. Hence, the operator $H'' - c\,Q''$ has a negative, infinite-dimensional spectral space. This fact fails to satisfy one of the basic assumptions [11] on $H - c\,Q$ to be used for stability analysis. Therefore, the Evans function becomes our choice to investigate stability of solitary waves in this paper.

3 The eigenvalue problem and the Evans function

The Evans function is defined by using solutions of an eigenvalue problem obtained from linearization of the underlying system about a travelling wave solution. We use the standard expression

$$\tilde{w} = w_c(x-ct) + e^{\lambda t}w(x-ct), \qquad \tilde{\eta} = \eta_c(x-ct) + e^{\lambda t}\eta(x-ct)$$

to take this procedure, where (w_c, η_c) is a solitary wave in the form of (2.1). Then the eigenvalue problem may be written as a system of the following two ordinary differential equations.

$$\begin{aligned} \lambda w &= \big(J_1 w + J_2 w' + J_3 w'' + J_4 \eta + J_5 \eta' + J_6 \eta''\big)', \\ \lambda \eta &= c\,\eta' - w', \end{aligned} \tag{3.1}$$

where $'$ represents the derivative with respect to ξ with $\xi = x - ct$, and J_k's are functions of the solitary wave and its derivatives such that

$$\begin{aligned} J_1 &= c - \frac{2w_c}{\eta_c} - \frac{\lambda}{3}\eta_c\eta_c' - \frac{2}{3}\eta_c' w_c' - \frac{2}{3} w_c \eta_c'' + \frac{2}{3}\eta_c w_c'' + \frac{2w_c(\eta_c')^2}{3\eta_c}, \\ J_2 &= \frac{\lambda}{3}\eta_c^2 + \frac{c\,\eta_c\eta_c'}{3} - \frac{2\eta_c' w_c}{3}, \qquad J_3 = \frac{2\eta_c w_c}{3} - \frac{c}{3}\eta_c^2, \\ J_4 &= -\frac{2c\,\eta_c w_c''}{3} + \frac{w_c^2}{\eta_c^2} - \eta_c + \frac{c}{3}\eta_c' w_c' + \frac{2}{3} w_c w_c'' - \frac{w_c^2(\eta_c')^2}{3\eta_c^2}, \\ J_5 &= \frac{c\eta_c w_c'}{3} - \frac{2}{3} w_c w_c' + \frac{2w_c^2\eta_c'}{3\eta_c}, \qquad J_6 = -\frac{1}{3} w_c^2. \end{aligned}$$

Equivalently, the eigenvalue problem may be expressed as a system of first order equations

$$Y' = AY, \tag{3.2}$$

where

$$Y = \begin{pmatrix} \eta \\ w \\ w' \\ w'' \end{pmatrix}, \qquad A = \begin{pmatrix} \frac{\lambda}{c} & 0 & \frac{1}{c} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{a_1}{\ell} & \frac{a_2}{\ell} & \frac{a_3}{\ell} & \frac{a_4}{\ell} \end{pmatrix},$$

with

$$\begin{aligned} a_1 &= J_4' + \frac{\lambda J_4}{c} + \frac{\lambda J_5'}{c} + \frac{\lambda^2 J_5}{c^2} + \frac{\lambda^2 J_6'}{c^2} + \frac{\lambda^3 J_6}{c^3}, \\ a_2 &= J_1' - \lambda, \qquad \ell = -J_3 - \frac{J_6}{c} = \frac{c}{3}, \\ a_3 &= J_1 + J_2' + \frac{J_4}{c} + \frac{J_5'}{c} + \frac{\lambda J_5}{c^2} + \frac{\lambda J_6'}{c^2} + \frac{\lambda^2 J_6}{c^3}, \\ a_4 &= J_2 + \frac{J_5}{c} + \frac{\lambda J_6}{c^2}. \end{aligned}$$

It follows from Weyl's essential spectral theorem again that the essential spectrum of the system $Y' = AY$ is the same as that of its asymptotic system $Y' = A^\infty Y$, where A^∞ is the limit of A, *i.e.*

$$A^\infty = \lim_{|x|\to\infty} A = \begin{pmatrix} \frac{\lambda}{c} & 0 & \frac{1}{c} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{3\lambda}{c^2} & -\frac{3\lambda}{c} & \frac{3(c^2-1)}{c^2} & \frac{\lambda}{c} \end{pmatrix}.$$

Since the characteristic polynomial $f(z)$ of A^∞ takes the form

$$f(z) = z^4 - \frac{2\lambda}{c} z + \Big(\frac{\lambda^2}{c^2} - \frac{3(c^2-1)}{c^2}\Big) z^2 + \frac{6\lambda}{c} z - \frac{3\lambda^2}{c^2} = (z^2 - 3)\big(z - \frac{\lambda}{c}\big)^2 + \frac{3z^2}{c^2},$$

it follows that f has a pure imaginary root if and only if λ is pure imaginary. Hence, the essential spectrum of A coincides with the imaginary axis in the λ-plane. In addition to its essential spectrum, the system also has an eigenvalue at $\lambda = 0$ with a one-dimensional eigenspace spanned by the eigenvector $Y_c^T = (\eta_c', w_c', w_c'', w_c''')$. This is a matter of fact for Hamiltonian systems that are translation invariant. We shall use it to find multiplicity of the zero for the Evans function at $\lambda = 0$.

To detect whether (3.2) has other eigenvalues, especially, those values lying on the right-half of the λ-plane, we now turn to the technique of the Evans function. It follows from [13] that the Evans function may be defined in a domain, Ω_λ, contained in the λ-plane such that the matrix A satisfies the following conditions.

H1. A is continuous in (ξ, λ), and analytic in λ for fixed ξ.

H2. $A \to A^\infty$ as $|\xi| \to \infty$ uniformly for λ in any compact set of Ω_λ.

H3. The integral $\int_{\mathbb{R}} |A - A^\infty| \, d\xi$ converges for all λ in Ω_λ, uniformly on compact sets.

H4. A^∞ has a unique, simple eigenvalue of smallest real part.

Since solitary wave solutions of the Green Naghdi equations are real analytic functions and decay exponentially at infinity, A satisfies the first three hypotheses for any complex value λ and uniformly in any compact set of the λ-plane. It also follows from [13] on the characteristic polynomial f that when $\Re\lambda \geq 0$, f has a unique and simple zero of smallest negative real part, denoted by $\mu(\lambda)$, and all other roots have nonnegative real parts. Hence, the right-half complex plane is contained in Ω_λ. The Evans function then may be formed by the dot product of the solutions Y and Z of (3.2) and its adjoint system, respectively, possibly with a scaling. In addition, Y and Z satisfy the conditions

$$\lim_{\xi \to \infty} e^{-\mu\xi} Y(\xi) \to v, \qquad \lim_{\xi \to -\infty} e^{\mu\xi} Z(\xi) \to w,$$

where v is an eigenvector of the eigenvalue μ of A and w is an eigenvector of $-\mu$ of $-A^T$, *i.e.* Y and Z lie in the stable and unstable manifolds of the systems $Y' = AY$ and $Z' = -A^T Z$, respectively. The Evans function $D = Z^T Y$ has a zero at λ if and only if Y and Z are homoclinic orbits in the two systems. In the next theorem, we show that eigenvalues of the system (3.2) appear symmetrically about the real and imaginary axes and therefore, it is sufficient to study the Evans function on the first quadrant of the complex plane.

Theorem 3.1 *Suppose that λ is an eigenvalue of the problem* (3.2). *Then $-\lambda$ and $\pm\overline{\lambda}$ are also eigenvalues of* (3.2)*, having the same geometric multiplicity as λ.*

Remark. It is worth noticing that if σ_k, for $k = 1, 2, 3, 4$, are roots of the characteristic polynomial f at λ, then $-\bar{\sigma}_k$'s are roots of f at $-\overline{\lambda}$. Especially, when the solution Y of (3.2) lies in the stable manifold at λ with $\Re\lambda \geq 0$, and $e^{-\mu\xi} Y \to v$ as $\xi \to \infty$, the solution V defined in the above theorem lies in the unstable manifold of the system (3.2) at $-\overline{\lambda}$ with $e^{\overline{\mu}\xi} V \to \overline{\tilde{v}}$ when $\xi \to -\infty$, where $\tilde{v} = (v_1, v_2, -v_3, v_4)^T$. Even though the Evans function D may be defined to be its analytic continuation while λ is going across the imaginary axis from the right hand complex plane, and V is not used to define the continuation for λ with $\Re\lambda < 0$ [13], the relations between solutions of (3.2) at $\pm\lambda$ and at the complex conjugate $\pm\overline{\lambda}$ show that one only needs to study the Evans function on the first quadrant to determine whether (3.2) has other zeros besides $\lambda = 0$. To conclude this section, now we show computations for the algebraic multiplicity of the eigenvalue $\lambda = 0$.

When $\lambda = 0$, one may integrate the system (3.1) once and use the relation between w and η to obtain the second order equation

$$\big(J_1 + J_4/c\big)w + \big(J_2 + J_5/c\big)w' + (J_3 + J_6/c)w'' = 0.$$

Then we find a solution of its adjoint problem

$$(J_1 + J_4/c)v - \big((J_2 + J_5/c)v\big)' + \big((J_3 + J_6/c)v\big)'' = 0$$

as $v = w_c'/\eta_c^2$. v is the derivative of the function $f_c = w_c/\eta_c$ from which a bounded solution $Z_c^T = (Z_1, Z_2, Z_3, Z_4)$ of the adjoint problem $Z' = -A^T Z$ of (3.2) may be formed so that

$$Z_1' = J_4' f_c, \quad Z_2' = J_1' f_c, \quad Z_3 = \big(J_2 + J_5/c\big) f_c + \ell f_c', \quad \text{and} \quad Z_4 = -\ell f_c.$$

It follows that

$$\begin{aligned} Z_1 &= \frac{w_c^2}{2c^3\eta_c}\big(-c^2(2c^2-1) + 4cw_c + 2w_c^2\big), \\ Z_2 &= -\frac{w_c^2}{c^2}, \quad Z_3 = \frac{(2w_c + c)w_c'}{3\eta_c^2}, \quad Z_4 = -\frac{cw_c}{3\eta_c}. \end{aligned}$$

Substituting Y_c and Z_c into the identity [13]

$$D'(0) = -\int_{\mathbb{R}} Z^T \left.\frac{\partial A}{\partial \lambda}\right|_{\lambda=0} Y\,dx$$

yields that the derivative of the Evans function at $\lambda = 0$ vanishes. The second derivative $D''(0)$ of the Evans function may be computed by using the expression

$$D''(0) = -\int_{\mathbb{R}} \left(Z_\lambda^T \left.\frac{\partial A}{\partial \lambda}\right|_{\lambda=0} Y + Z^T \left.\frac{\partial A}{\partial \lambda}\right|_{\lambda=0} Y_\lambda + Z^T \left.\frac{\partial^2 A}{\partial \lambda^2}\right|_{\lambda=0} Y \right) dx, \tag{3.3}$$

where Z_λ and Y_λ denote derivatives of Y and $Z$with respect λ. To compute Y_λ, we observe that if (w, η) is a solution in the stable manifold of (3.1) at $\lambda = 0$, $(w, \eta) = (w_c', \eta_c')$ and it satisfies equations

$$\begin{aligned} &J_1 w_\lambda + J_2 w_\lambda' + J_3 w_\lambda'' + J_4 \eta_\lambda + J_5 \eta_\lambda' + J_6 \eta_\lambda'' = w_c - \frac{\partial J_1}{\partial \lambda} w_c' - \frac{\partial J_2}{\partial \lambda} w_c'', \\ &\eta_c - 1 - c\eta_\lambda + w_\lambda = 0. \end{aligned}$$

Substituting $\eta_\lambda = (\eta_c - 1)/c + w_\lambda/c$ into the first equation, one obtains a non-homogeneous equation to solves for w_λ by using the fact that the corresponding homogeneous equation has a solution w_c' and the other solution independent of w_c' may be computed by means of reduction of order. Then the method of variation of parameters leads to the result

$$w_\lambda = \frac{w_c}{c^2(c^2-1)}\big(c - 3c^3 + \sqrt{3(c^2-1)}\, x \tanh \frac{\sqrt{3(c^2-1)}\, x}{2}\big).$$

Therefore, $Y_\lambda^T = ((\eta_c - 1)/c + w_\lambda/c,\ w_\lambda, w_\lambda', w_\lambda'')$.

Z_λ may be computed in a similar way. Because the adjoint system of (3.1) takes the form

$$\begin{aligned} J_1 f' - (J_2 f')' + (J_3 f')'' + \lambda f + g' = 0, \\ J_4 f' - (J_5 f')' + (J_6 f')'' - \lambda g - cg' = 0, \end{aligned}$$

at $\lambda = 0$, the derivative (f_λ, g_λ) of its solution (f, g) with respect to λ satisfies the equations

$$\begin{aligned} &J_1 f_\lambda' - (J_2 f_\lambda')' + (J_3 f_\lambda')'' + f + g_\lambda' + \frac{\partial J_1}{\partial \lambda} f' - \big(\frac{\partial J_2}{\partial \lambda} f'\big)' = 0, \\ &J_4 f_\lambda' - (J_5 f_\lambda')' + (J_6 f_\lambda')'' - g - cg_\lambda' = 0. \end{aligned}$$

When $(f, g) = (f_c, g_c)$ with $f_c = \frac{w_c}{\eta_c}$ and $g_c' = -J_1 f_c' + (J_2 f_c')' - (J_3 f_c')''$, *i.e.* (f, g) is a homoclinic orbit of the adjoint problem, one may combine the above two equations to obtain the non-homogeneous equation

$$(J_1 + J_4/c)f_\lambda' - ((J_2 + J_5/c)f_\lambda')' + ((J_3 + J_6/c)f_\lambda')'' + f_c - g_c/c + \frac{\partial J_1}{\partial \lambda} f_c' - (\frac{\partial J_2}{\partial \lambda} f_c')' = 0.$$

Therefore, under the boundary condition $f_\lambda, g_\lambda \to 0$ as $x \to -\infty$, one may apply the methods of reduction of order and variation of parameters to the above equation and obtain the solution

$$f_\lambda = \frac{1}{3\sqrt{c^2-1}\,\eta_c}\Big(\frac{3xw_c}{c\sqrt{c^2-1}} + 2\sqrt{3}\,c(2c^2-1)(1+\tanh y) + 2\sqrt{3}\,(2c^2-1)w_c\Big),$$

and consequently,

$$\begin{aligned} g_\lambda = &-\frac{2c}{\sqrt{3}} \ln \frac{(c+\sqrt{c^2-1}\,\tanh y)(c+\sqrt{c^2-1}\,)}{(c-\sqrt{c^2-1}\,\tanh y)(c-\sqrt{c^2-1}\,)} + \frac{2c^2(c^2-2)}{\sqrt{3(c^2-1)}}\Big(1+\frac{\tanh y}{\eta_c}\Big) + \\ &-\frac{2}{3(c^2-1)\eta_c}\left(-\frac{3(c^2-1)w_c^2 w_c'}{c^2} + (8c^4-7c^2+4)\frac{w_c w_c'}{2c} + (2c^4-c^2-2)\frac{w_c'}{2}\right) + \\ &-\frac{xw_c}{2c^2(c^2-1)\eta_c}\big(2 + (4c^2-3)w_c/c - 5w_c^2/c^2\big), \end{aligned}$$

where $y = \sqrt{3(c^2-1)}\,x/(2c)$. Hence, $Z_\lambda = (Z_{1\lambda}, Z_{2\lambda}, Z_{3\lambda}, Z_{4\lambda})$ may be obtained by taking the derivative with respect to λ on both sides of the following identities

$$\begin{aligned} Z_1 &= -cg + \big(J_4 + \frac{\lambda}{c}J_5 + \frac{\lambda^2}{c^2}J_6\big)f + (J_6 f')' - \big(J_5 + \frac{\lambda}{c}J_6\big)f', \\ Z_2 &= g + J_1 f - J_2 f' + (J_3 f')', \\ Z_3 &= \ell f' + \big(J_2 + \frac{1}{c}J_5 + \frac{\lambda}{c^2}J_6\big)f, \\ Z_4 &= -\ell f, \end{aligned}$$

valid for any λ, and substituting $\lambda = 0$, $(f, g) = (f_c, g_c)$ and (f_λ, g_λ) as computed above into the resulting equations. Using the above computations, we now state the result on $D''(0)$ in the following theorem.

Theorem 3.2 *At $\lambda = 0$, the Evans function of the eigenvalue problem* (3.1) *has a zero of multiplicity two, and thus the eigenvalue $\lambda = 0$ has an algebraic multiplicity equal to two.*

Proof. Substituting $Y = Y_c$, $Z = Z_c$, Y_λ and Z_λ into the equation (3.3) and taking integration, we obtain the following identity.

$$D''(0) = \frac{16\sqrt{1+b^2}\,\big(2b\sqrt{1+b^2} - \ln(b+\sqrt{1+b^2}\,)\big)\mathrm{sign}\, c}{\sqrt{3}},$$

where $b = \sqrt{c^2-1}$. Hence, $|D''(0)| > 0$ for any $b > 0$. It follows from Theorem 4 [13] that one may normalize D at infinity by choosing the scaling factor ϱ as the scalar product of the vectors $v_0 = \lim_{x\to-\infty} e^{\mu x} Z$ and $w_0 = \lim_{x\to\infty} e^{-\mu x} Y$, where $\mu = \mu(\lambda)$ is the eigenvalue of the smallest real part of the asymptotic matrix A^∞. When $\lambda = 0$, $\mu = -\sqrt{3(c^2-1)}/|c|$, $v_0 =$

$-\frac{4c^2(c^2-1)}{3}(0,0,\mu,1)^T$ and $w_0 = 4\mu(c^2-1)(1,c,c\mu,c\mu^2)^T$. Therefore, $\varrho = 32(c^2-1)^{9/2}\operatorname{sign} c = 32b^9 \operatorname{sign} c$, and the second derivative of the normalized Evans function $\tilde{D}$ becomes

$$\tilde{D}''(0) = \frac{\sqrt{1+b^2}\,\big(2b\sqrt{1+b^2} - \ln(b+\sqrt{1+b^2}\,)\big)}{2\sqrt{3}\,b^9}.$$

The second conclusion follows from the fact [2] that the multiplicity of a zero λ_0 of the Evans function is equivalent to the algebraic multiplicity of the eigenvalue λ_0, and $\tilde{D} > 0$ for any b with $b = \sqrt{c^2-1}$.

4 Asymptotic behavior of the Evans function

It has been shown that if the system (3.1) satisfies conditions of Proposition 1.17 in [14], then the normalized Evans function approaches one as $|\lambda| \to \infty$. However, there is a difficulty to apply the Proposition to the Green-Naghdi equations. If (3.1) is changed to a system of first order equations, then coefficients of the reduced system (3.2) contains powers of λ up to the third order. As a result, the integral $\int_{\mathbb{R}} |F| ds$ as defined in [14] is not bounded for large λ. It has also been pointed out in [14], the conditions in the proposition are only sufficient. Here, we shall use the Hamiltonian structure of the Green Naghdi equations to decompose differential operators in (3.1), and use the result to show that when $\gamma = \sqrt{1-c^{-2}}$ is sufficiently small, the Evans function has no zero for any sufficiently large $|\lambda|$.

Lemma 4.1 *Let (w,η) be a solution of the system (3.1). Then there are functions u, v, Δ_1 and Δ_5, depending on the solitary wave solution (w_c,η_c) and λ, and an integer $N > 0$ such that whenever $|\lambda| \geq N$, the function η' can be expressed as*

$$\eta' = \frac{-1}{\lambda\rho}\big(s_1 w'' + (1-s_1)w'' - \frac{s_2 b w'}{\lambda} - \frac{(1-s_2)bw'}{\lambda}\big) + \frac{1}{\lambda\rho}\big(c\eta'' - \frac{s_3 c b \eta'}{\lambda} + b\eta\big),$$

the linear operator $\mathcal{E}_\lambda$, defined by

$$\mathcal{E}_\lambda f = -\big(a_2 + \frac{s_1}{\lambda^2\rho}\big)f'' - \big(a_1 - \frac{s_2 b}{\lambda^3\rho} + \frac{u}{\lambda}\big)f' + \big(a_3 + 1 - \frac{v}{\lambda}\big)f,$$

is invertible on the function space $\{f;\ \sup_{|x|<\infty} |e^{a|x|}f^{(j)}(x)| < \infty, j = 0,1,2\}$ for any fixed $a \geq 0$, and the first equation of the system (3.1) can be decomposed as

$$\begin{aligned}\lambda\mathcal{E}_\lambda w = &\mathcal{E}_\lambda\Big(\frac{ca_2 - \frac{2w_c\eta_c}{3}}{a_2 + \frac{s_1}{\lambda^2\rho}}w' + \frac{w_c^2}{3(a_2 + \frac{s_1}{\lambda^2\rho})}\eta' + \frac{c}{\lambda\rho\big(a_2 + \frac{s_1}{\lambda^2\rho}\big)}\eta\Big) + \\ &- \mathcal{E}_\lambda\Big(\frac{\Delta_1 + J_2^* + J_3'}{a_2 + \frac{s_1}{\lambda^2\rho}}w + \frac{\Delta_5 + J_5 + J_6'}{a_2 + \frac{s_1}{\lambda^2\rho}}\eta\Big) + \frac{1}{\lambda}\mathcal{U},\end{aligned}$$

where $a_0 = \frac{1}{3}(\eta_c\eta_c')'$, $a_1 = \frac{1}{3}\eta_c\eta_c'$ and $a_2 = \frac{1}{3}\eta_c^2$, and the coefficients b, ρ and s_k, for $k = 1,2,3$, satisfy the equations

$$1 - s_1 = \frac{a_2 s_2 b\, c}{\lambda^2\big(a_2 + \frac{s_1}{\lambda^2\rho}\big)}, \qquad c - \frac{ca_2}{a_2 + \frac{s_1}{\lambda^2\rho}} = \frac{(1-s_2)b}{\lambda^2\rho}, \qquad b = \frac{-c}{a_2 + \frac{s_1}{\lambda^2\rho}},$$

$$s_3 = \frac{s_2}{\lambda^2\rho(a_2 + \frac{s_1}{\lambda^2\rho})}, \qquad \rho = 1 + \frac{(1-s_3)bc}{\lambda^2}.$$

In addition, $\mathcal{U}$ is a linear function of $y = (w, \eta)$, y' and y'', together with the expressions u, v, $\varDelta_1$ and $\varDelta_5$, satisfying the inequalities

$$|\mathcal{U}| \le M\gamma^2 e^{-\gamma|x|}(\|y\| + \|y'\| + \|y''\|), \quad |u| \le M\gamma^2 e^{-\gamma|x|}, \quad |v| \le M\gamma^2 e^{-\gamma|x|},$$
$$|\varDelta_1| \le M\gamma^2 e^{-\gamma|x|}, \quad |\varDelta_5| \le M\gamma^2 e^{-\gamma|x|}.$$

for some constant M independent of λ for any $|\lambda| \ge N$.

Using the above Lemma, one may estimate the characteristic polynomial of the constant matrix A^∞, and the asymptotic behavior of the Evans function as $|\lambda|$ is sufficiently large. We now demonstrate these results as the consequences of the Lemma.

Corollary 4.2 *For any $|\lambda|$ sufficiently large, the roots of the characteristic polynomial of A^∞ coincide with the associated characteristic roots of the differential operator $\mathcal{E}_\lambda^\infty$ and the matrix*

$$\mathcal{A}_1^\infty = \lambda \begin{pmatrix} \frac{1}{c} + \frac{3s_1^\infty}{\lambda^2 c\rho^\infty} & \frac{-3}{\lambda^2\rho^\infty} \\ \frac{1}{c^2} + \frac{3s_1^\infty}{\lambda^2 c^2\rho^\infty} & \frac{1}{c} - \frac{3}{\lambda^2 c\rho^\infty} \end{pmatrix}.$$

Theorem 4.3 *For any sufficiently large $|\lambda|$ with $\Re\lambda \ge 0$ and any sufficiently small $\gamma > 0$, the system* (3.1) *has no bounded non-trivial solutions decaying to zero at $x = \infty$, and thus the Evans function is non vanishing.*

5 The KdV approximation of the Green Naghdi equations

As has been pointed out in [9] that the KdV equations is an approximation of the Green-Naghdi equations. Let $\gamma^2 = 1 - c^{-2}$. One may apply the following transformations [14]

$$s = \gamma(x - ct), \quad \tau = c\gamma^3 t, \quad w = c\gamma^2 u, \quad \eta = 1 + \gamma^2 v$$

to the system (2.1), and the second order approximation $\eta = 1 + \gamma^2 v_1 + \gamma^4 v_2 + \cdots$ and $w = c(\gamma^2 u_1 + \gamma^4 u_2 + \cdots)$, for the small parameter $\gamma > 0$, yields the system

$$\begin{aligned} &v_{1s} - u_{1s} = 0, \\ &u_{1\tau} + v_{1\tau} - u_{1s} = -(u_1^2)_s - v_1 v_{1s} - \frac{1}{3}u_{1sss}. \end{aligned} \tag{5.1}$$

Under the conditions $u_1, v_1 \to 0$ as $|s| \to \infty$, we derived the KdV equation

$$u_{1\tau} - \frac{1}{2}u_{1s} + \frac{3}{4}(u_1^2)_s + \frac{1}{6}u_{1sss} = 0.$$

This approach also leads to a rescaling of the eigenvalue problem (3.1) so that one may obtain an equivalent system for which the linearized KdV equation about a solitary wave becomes its approximation. Correspondingly, we let $\lambda = c\gamma^3\Lambda$, $s = \gamma\xi$, $w = c\tilde{w}$, $w_c = c\gamma^2\tilde{w}_c$ and $\eta_c = 1 + \gamma^2\tilde{\eta}_c$. Then $\tilde{w}_c = \tilde{\eta}_c = c^2 \operatorname{sech}^2 \frac{\sqrt{3}x}{2}$. Substituting these transformations to the system (3.1) and dropping the accent ˜ for simplicity, we obtain the system

$$\begin{aligned} &\gamma^2\Lambda\eta - \eta' + w' = 0, \\ &\Lambda w + \Lambda\eta = (\tilde{J}_1 w + \gamma^2\tilde{J}_2 w' + \tilde{J}_3 w'' + \tilde{J}_4\eta + \gamma^2\tilde{J}_5\eta' + \gamma^4\tilde{J}_6\eta'')', \end{aligned} \tag{5.2}$$

where $\tilde{J}_k$'s are given functions expressed as

$$\begin{aligned}
\tilde{J}_1 &= -\frac{2\eta_c}{1+\gamma^2\eta_c} - \frac{\gamma^4\Lambda}{3}(1+\gamma^2\eta_c)\eta_c' - \frac{2\gamma^4}{3}(\eta_c')^2 - \frac{2\gamma^4}{3}\eta_c\eta_c'' + \\
&\quad + \frac{2\gamma^6\eta_c(\eta_c')^2}{3(1+\gamma^2\eta_c)} + \frac{2\gamma^2}{3}(1+\gamma^2\eta_c)\eta_c'', \\
\tilde{J}_2 &= \frac{\Lambda(1+\gamma^2\eta_c)^2}{3} + \frac{(1+\gamma^2\eta_c)\eta_c'}{3} - \frac{2\gamma^2\eta_c\eta_c'}{3}, \\
\tilde{J}_3 &= \frac{1}{3}(\gamma^4\eta_c^2 - 1), \\
\tilde{J}_4 &= 1 - (1-\gamma^2)\eta_c - \frac{2\gamma^2}{3}(1+\gamma^2\eta_c)\eta_c'' + \frac{\gamma^2\eta_c^2}{(1+\gamma^2\eta_c)^2} + \frac{\gamma^4}{3}(\eta_c')^2 + \\
&\quad + \frac{2\gamma^4}{3}\eta_c\eta_c'' - \frac{\gamma^8\eta_c^2(\eta_c')^2}{3(1+\gamma^2\eta_c)^2}, \\
\tilde{J}_5 &= \frac{(1+\gamma^2\eta_c)\eta_c'}{3} - \frac{2\gamma^2}{3}\eta_c\eta_c' + \frac{2\gamma^4\eta_c^2\eta_c'}{3(1+\gamma^2\eta_c)}, \qquad \tilde{J}_6 = -\frac{\eta_c^2}{3}.
\end{aligned}$$

When $\gamma = 0$, we obtain the system

$$\begin{aligned}
&\eta' - w' = 0 \\
&\Lambda w + \Lambda\eta = (-2\eta_c w - \frac{1}{3}w'' + \eta - \eta_c\eta)'.
\end{aligned}$$

Since we look for solutions decaying to zeros at infinity, the above system is equivalent to the equations

$$\eta = w, \qquad \Lambda w = (\frac{1}{2}w - \frac{3}{2}\eta_c w - \frac{1}{6}w'')'.$$

The second equation is the linearized KdV equation (5.1) about its solitary wave solution $\eta_{0c} = \operatorname{sech}^2 \frac{\sqrt{3}x}{2}$. It is also well known that the Evans function of the eigenvalue problem for the KdV equation takes the form

$$D = \left(\frac{\mu+\sqrt{3}}{\mu-\sqrt{3}}\right)^2,$$

where μ is the root of the smallest real part of the characteristic polynomial for the equation $\mu^3 - 3\mu + 3\Lambda = 0$. Therefore, the linearized KdV equation has only one eigenvalue at $\lambda = 0$ with an algebraic multiplicity two. Since in any bounded domain of Λ, (5.2) is a regular perturbation problem with the small parameter γ, solutions of (5.2) are convergent to those of linearized KdV equation when $\gamma \to 0$ and the Evans function of (5.2) has the same zeros as that of the KdV equation for any sufficiently small γ as well. On the other hand, on any bounded domain of λ, the problem (3.1) may be expressed as a regular perturbation of the equations

$$\lambda w = cw' - (1-\partial^2/3)^{-1}\eta, \qquad \lambda\eta = -w' + c\eta'$$

with respect to the parameter γ, and the operator

$$\begin{pmatrix} \lambda - c\partial & (1-\partial^2/3)^{-1} \\ \partial & \lambda - c\partial \end{pmatrix}$$

has a bounded inverse outside any neighbourhood of the origin $\lambda = 0$ with $\Re\lambda \geq 0$, and thus (3.1) has no eigenvalues when $\lambda \neq 0$ in this case. Combining Theorem 4.4 with results in this section, one may conclude that for any γ sufficiently small, the only eigenvalue of the problem (3.1) is $\lambda = 0$.

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